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Second Quantization of the Dirac Field: Normal Modes in the Robertson–Walker Space-Time

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The quantization of the Dirac field in the context of the Robertson–Walker space-time is reconsidered in some of its constitutive elements. The particular solutions of the Dirac equation previously determined are used to construct the normal mode solutions in the case of flat, closed, and open space-time. The procedure is based on a general standard definition of inner product between solutions of the Dirac equation that is applied by making use of an integral property of the separated time equation. The open-space case requires the recurrence relations of functions associated to solutions of the Dirac equation.

1. INTRODUCTION

The Dirac equation can be formulated in a curved space time by means of the spinor calculus (Penrose and Rindler, 1990). The formulation can be done in general and it does not suffer from the limitations of similar equations for higher spin values (Penrose and Rindler, 1990; Buchdahl, 1962; Wünsch, 1978, 1979; Illge, 1993).

The second quantization of the Dirac field in curved space-time is also a subject that has received great attention (Parker, 1971; Unruh, 1974; Ford, 1976) and it can be considered as a special case of the quantization of field theory in curved space-time (Birrell and Davies, 1982; Fulling, 1989). The quantization procedure is in general not compatible with the principle of covariance in general relativity. This means that, in contrast to the Minkowski space case, it depends on the choice of the coordinate system (Birrell and Davies, 1982; Fulling, 1973). However, as in the flat-space case, it is based on the knowledge of the normal modes associated to the solutions of the

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field equation, their existence being ensured on general grounds. Explicit mode solutions have been considered for some specific examples of metrics such as, for example, that of an expanding universe with Euclidean 3-space (Parker, 1971), that of the Kerr metric for the neutrino case (Unruh, 1974), that of the static Einstein universe (Ford, 1976), and for field equations other than the Dirac one (Ford, 1976; Birrell and Davies, 1982; Fulling, 1973, 1989; Parker, 1969; Parker and Fulling, 1974; Hu, 1974).

The object of this paper is to determine a complete set of orthonormal modes of the Dirac equation in the Robertson–Walker space-time. The physical situation is of interest not only because the Robertson–Walker metric is the basis of the standard cosmology (Kolb and Turner, 1990), but also because such a metric has an explicit time dependence that makes the corresponding Dirac equation not directly integrable in the Newman–Penrose formalism by the usual separation method.

The normalization procedure is applied to the solutions determined in a previous paper (Zecca, 1996). By using the Newman–Penrose formalism (Newman and Penrose, 1962), the problem of the solution of the Dirac equation has been there reduced to the solution of separated equations in the different coordinates, giving rise to a complete set of nonfactorized solutions.

The inner product in the space of the solutions of the Dirac equation, which can be defined in a standard way on general grounds, is explicitly calculated in the case under consideration by taking into account a formal property of the separated time equation to which the time evolution was reduced.

The cases of the closed and the open universe are directly treated by elaborating the particular solutions. The open-universe case requires the study of recurrence, differential, and integral properties of functions associated to solutions of the Dirac equation.

2. STATEMENT OF THE PROBLEM

The problem is discussed directly in the Robertson–Walker space-time whose line element is written as

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \qquad (a = 0, \pm 1) \quad (1)$$

Besides the standard correspondence between spinors and tensors provided by Infeld–van der Waerden quantities $\sigma^a_{AA'}$ it is also understood the use of the specific null tetrad frame on which are based the results of Zecca, (1996).

The formulation of the Dirac equation is written in spinor form as

$$\nabla_{AA'} P^A + i\mu_\star \overline{Q}_{A'} = 0$$
$$\nabla_{AA'} Q^A + i\mu \overline{P}_{A'} = 0 \tag{2}$$

where $\nabla_{AA'}$ are the covariant spinor derivatives and $\mu_\star\sqrt{2}$ the mass of the particle Chandrasekhar, 1983). In correspondence to the solutions $\phi \leftrightarrow (P, Q)$, $\psi \leftrightarrow (U, V)$, one can define the spinor

$$J^{AA'}(\phi, \psi) = P^A \overline{U}^{A'} + V^A \overline{Q}^{A'} \tag{3}$$

which is divergence-free, $\nabla_{AA'} J^{AA'} \equiv \nabla_\alpha J^\alpha = 0$, as a consequence of equation (2).

An inner product between the solutions of the Dirac equation can be defined by setting

$$(\phi, \psi) \equiv \int_\Sigma J_\alpha(\phi, \psi)(-g_\Sigma(x))^{1/2} n^\alpha \, d\Sigma \tag{4}$$

$$= \int_{t=t_0} d_3x(-g_{t_0})^{1/2} \sigma^t_{AA'} J^{AA'}(\phi, \psi) \tag{5}$$

$$= \frac{1}{2} \int_{t=t_0} d_3x(-g_{t_0})^{1/2} (P^0\overline{U}^0 + P^1\overline{U}^1 + V^0\overline{Q}^0 + V^1\overline{Q}^1 \tag{6}$$

because, by Gauss' theorem (Hawking and Ellis, 1973), the expression (4) is independent of the spacelike Cauchy hypersurface Σ of volume element $d\Sigma$, n^α being a future-directed unit vector orthogonal to $d\Sigma$. Therefore the product can be calculated by the more tractable expression (5) and finally the simplest one (6), because in the null-tetrad frame under consideration the generalized Pauli matrix $\sigma^t_{AA'}$ has the form $\sigma^t_{AA'} = \frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ (compare also with Montaldi and Zecca, 1994)

Solutions of equation (2) can be easily obtained from the results of Zecca (1996), by a suitable choice of a constant of integration. Without loss of generality they can be put into the form

$$P^0_{klm} = \frac{S^{(1)}_{lm}(\theta, \varphi)}{2rR(t)} H^+(r, t) \left\{ A_{lk}(s) T_k(\tau) + \left[\frac{2\lambda}{r} A_{lk}(s) - A'_{lk}(s) \right] \int_0^\tau d\hat{\tau}\, T_k(\hat{\tau}) \right\}$$

$$P^1_{klm} = \frac{S^{(2)}_{lm}(\theta, \varphi)}{2rR(t)} H^+(r, t) \left\{ -A_{lk}(s) T_k(\tau) + \left[\frac{2\lambda}{r} A_{lk}(s) - A'_{lk}(s) \right] \int_0^\tau d\hat{\tau}\, T_k(\hat{\tau}) \right\}$$

$$\overline{Q}^1_{klm} = \frac{S^{(1)}_{lm}(\theta, \varphi)}{2rR(t)} H^+(r, t) \left\{ -A_{lk}(s) T_k(\tau) + \left[\frac{2\lambda}{r} A_{lk}(s) - A'_{lk}(s) \right] \int_0^\tau d\hat{\tau}\, T_k(\hat{\tau}) \right\}$$

$$\overline{Q}^0_{klm} = -\frac{S^{(2)}_{lm}(\theta, \varphi)}{2rR(t)} H^+(r, t)\left\{ A_{lk}(s)T_k(\tau) + \left[\frac{2\lambda}{r} A_{lk}(s) - A'_{lk}(s)\right] \int_0^\tau d\hat{\tau}\, T_k(\hat{\tau})\right\} \tag{7}$$

The angular functions appearing in (7) are of the form $S^{(i)}_{lm} = S_{ilm}(\theta, \varphi) \exp(im\varphi)$, $(i = 1, 2)$, where $S_{ilm}(\theta, \varphi)$ are solutions of an eigenvalue problem originated by the solution of the angular part of (2). They are essentially the Jacobi polynomials for $|m| \geq 1$ with $\lambda^2 = (l + 1/2)^2$, $l = |m|, |m| + 1, |m| + 2, \ldots$, and are essentially the Tchebicheff polynomials for $m = 0$ with $\lambda^2 = (l + 1)^2$, $l = 0, 1, 2, \ldots$ (Zecca, 1996; Montaldi and Zecca, 1994). The angular functions are assumed to satisfy the normalization condition

$$\int d\Omega\, S^{(i)}_{lm}(\theta, \varphi) S^{(i)}_{l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'} \qquad (i = 1, 2) \tag{8}$$

The function H^+ is connected to a particular integral of (2) and has the form

$$H^+(r, t) = \frac{K^+}{R^{1/2}(t)} \left(\frac{1 + \sqrt{1 - ar^2}}{r}\right)^\lambda exp(i\mu_\star\sqrt{2}t) \qquad (a = 0, \pm 1) \tag{9}$$

while the conformal time parameter τ and the spatial parameter s are defined by

$$\tau(t) = \int_0^t \frac{dt'}{R(t')}, \qquad s(r) = \int_0^r \frac{dr'}{\sqrt{1 - ar'^2}} \qquad (a = 0, \pm 1) \tag{10}$$

By these variables the functions $A_{lk}(s)$ come out to be solutions of radial equations and their explicit expression, as given in Zecca, (1996), will be used in the following.

The function $T_k(\tau)$ is a solution of the separated time equation in the conformal time parameter (Zecca, 1996)

$$T'' + 2\sqrt{2}i\mu_\star R(\tau)T' + (2i\sqrt{2}\mu_\star R'(\tau) + k^2)T = 0 \tag{11}$$

where k^2 is a separation constant. The solution of equation (11) is in general difficult. It depends on the dynamical evolution of the cosmological background. In the case of the standard cosmology where $R(t)$ is given in a parametric form, the function R can be easily given as a function of the conformal time parameter τ (Zecca, 1997).

With regard to the concrete application of the normalization condition (6) to the solution (7), we remark that a solution $T_k(\tau)$ of equation (11) can always be chosen so that

$$|T_k(\tau)|^2 + k^2 \left| \int_0^\tau T_k(\hat\tau)\, d\hat\tau \right|^2 = 1 \tag{12}$$

as follows by a direct derivation of equation (12) with respect to τ and by the use of the expression $\int_0^\tau T_k d\hat\tau$, which follows by a first integration of (11).

3. NORMAL MODE SOLUTIONS

The normalization procedure will be applied separately for $a = 0, \pm 1$.

3.1. The Flat Case $a = 0$

From the definition (10) one has $r = s$ and from the result of . . . (Zecca, 1996)

$$A_{lk}(r) = (2ikr)^{2\lambda} e^{-ikr} \Phi(\lambda;\, 2\lambda;\, 2ikr) \tag{13}$$

where Φ is the confluent hypergeometric function. Therefore from (7), from the recurrence relations of the hypergeometric function, and from its expression in terms of the Bessel functions one has

$$\begin{aligned}
P^0_{klm} &= S^{(1)}_{lm}(\theta, \varphi) \frac{\exp(i\mu_\star\sqrt{2}t)}{R^{3/2}(t)} \sqrt{\frac{k}{2r}} \left(J_{\lambda-1/2}(kr) T_k - ikJ_{\lambda+3/2}(kr) \int_0^\tau d\hat\tau\, T_k \right) \\
P^1_{klm} &= S^{(2)}_{lm}(\theta, \varphi) \frac{\exp(i\mu_\star\sqrt{2}t)}{R^{3/2}(t)} \sqrt{\frac{k}{2r}} \left(-J_{\lambda-1/2}(kr) T_k - ikJ_{\lambda+3/2}(kr) \int_0^\tau d\hat\tau\, T_k \right) \\
\bar{Q}^1_{klm} &= S^{(1)}_{lm}(\theta, \varphi) \frac{\exp(i\mu_\star\sqrt{2}t)}{R^{3/2}(t)} \sqrt{\frac{k}{2r}} \left(-J_{\lambda-1/2}(kr) T_k - ikJ_{\lambda+3/2}(kr) \int_0^\tau d\hat\tau\, T_k \right) \\
\bar{Q}^0_{klm} &= S^{(2)}_{lm}(\theta, \varphi) \frac{\exp(i\mu_\star\sqrt{2}t)}{R^{3/2}(t)} \sqrt{\frac{k}{2r}} \left(-J_{\lambda-1/2}(kr) T_k + ikJ_{\lambda+3/2}(kr) \int_0^\tau d\hat\tau\, T_k \right)
\end{aligned} \tag{14}$$

where $(k/2)^{1/2}$ is inserted for later convenience. From equation (6) it follows that, denoting

$\psi_{klm} \leftrightarrow (P_{klm}, Q_{klm})$,

$$\begin{aligned}
(\psi_{klm}, \psi_{k'l'm'}) = k\delta_{ll'}\delta_{mm'} \Bigg\{ & T_k \bar{T}_{k'} \int_0^\infty dr\, rJ_{\lambda-1/2}(kr) J_{\lambda-1/2}(k'r) \\
& + kk' \int_0^\infty dr\, rJ_{\lambda+3/2}(kr) J_{\lambda+3/2}(k'r) \int_0^\tau d\hat\tau\, T_k \int_0^\tau d\hat\tau\, \bar{T}_{k'} \Bigg\}
\end{aligned}$$

$$= k\delta_{ll'}\delta_{mm'}\left[\,|T_k|^2 + k^2\left|\int_0^\tau T_k\, d\hat{\tau}\right|^2\,\right]\frac{\delta(k-k')}{k}$$
$$= \delta_{ll'}\delta_{mm'}\delta(k - kl) \tag{15}$$

where the Bessel function closure equation (Arfken and Weber, 1995) has been used together with the normalization assumption (12).

3.2. The Closed Case $a = 1$

We have $r = \sin s$ $(0 \le s \le \pi)$ and we choose $A_{nl}(s) = x^{1/2}(1 - x)^{\lambda}$ $F(-2n, 2\lambda + 2n + 1; \lambda + 1/2; 1 - x)$, where $x = (\cos s + 1)/2$ (Zecca, 1996), F being the hypergeometric function (Hawking and Ellis, 1973). [There is a *mistake* in Zecca, (1996): the A_{nl} function we have assumed *does not* satisfy the constraint required there.] By using the differential formula and the recurrence relations for the Jacobi polynomials (Abramovitz and Stegun, 1970), one gets from equations (7) and (9)

$$P^0_{nlm} = C\frac{S^{(1)}_{lm}(\theta,\varphi)}{rR^{3/2}(t)}\exp(i\mu_\star\sqrt{2}t)[x(1-x)]^{\lambda/2}\Big\{T_n x^{1/2}P^{(\lambda-1/2,\lambda+1/2)}_{2n}(2x-1)$$
$$+ k_n(1-x)^{1/2}P^{(\lambda+1/2,\lambda-1/2)}_{2n}(2x-1)\int_0^\tau d\hat{\tau}\, T_n\Big\}$$
$$P^1_{nlm} = C\frac{S^{(2)}_{lm}(\theta,\varphi)}{rR^{3/2}(t)}\exp(i\mu_\star\sqrt{2}t)[x(1-x)]^{\lambda/2}\Big\{-T_n x^{1/2}P^{(\lambda-1/2,\lambda+1/2)}_{2n}(2x-1)$$
$$+ k_n(1-x)^{1/2}P^{(\lambda+1/2,\lambda-1/2)}_{2n}(2x-1)\int_0^\tau d\hat{\tau}\, T_n\Big\}$$
$$\overline{Q}^1_{nlm} = C\frac{S^{(1)}_{lm}(\theta,\varphi)}{rR^{3/2}(t)}\exp(i\mu_\star\sqrt{2}t)[x(1-x)]^{\lambda/2}\Big\{-T_n x^{1/2}P^{(\lambda-1/2,\lambda+1/2)}_{2n}(2x-1)$$
$$+ k_n(1-x)^{1/2}P^{(\lambda+1/2,\lambda-1/2)}_{2n}(2x-1)\int_0^\tau d\hat{\tau}\, T_n\Big\}$$
$$\overline{Q}^0_{nlm} = C\frac{S^{(1)}_{lm}(\theta,\varphi)}{rR^{3/2}(t)}\exp(i\mu_\star\sqrt{2}t)[x(1-x)]^{\lambda/2}\Big\{-T_n x^{1/2}\mathrm{P}^{(\lambda-1/2,\lambda+1/2)}_{2n}(2x-1)$$
$$- k_n(1-x)^{1/2}P^{(\lambda+1/2,\lambda-1/2)}_{2n}(2x-1)\int_0^\tau d\hat{\tau}\, T_n\Big\} \tag{16}$$

where $x = (\cos s + 1)/2$ and we set

$$k_n = 2\lambda + 2n + 1, \qquad C = \frac{((2n)!)^{1/2}\Gamma^{1/2}(k_n + \lambda + 1/2)}{\Gamma(k_n)} \tag{17}$$

Therefore, denoting $\psi_{nlm} \leftrightarrow (P_{nlm}, Q_{nlm})$, we have

$$\begin{aligned}
&(\psi_{nlm}, \psi_{n'l'm'}) \\
&= \frac{|C|^2}{2^\lambda}\delta_{ll'}\delta_{mm'}\Bigg\{T_n\overline{T}_{n'}\int_{-1}^{1} dy\,(1-y)^{\lambda-1/2}(1+y)^{\lambda+1/2} \\
&\quad \times P_{2n}^{(\lambda-1/2,\lambda+1/2)}(y)P_{2n}^{(\lambda-1/2,\lambda+1/2)}(y) \\
&\quad + \int_0^\tau d\tau\, T_n \int_0^\tau d\tau\, \overline{T}_{n'} \int_{-1}^{1} dy\,(1-y)^{\lambda+1/2}(1+y)^{\lambda-1/2} \\
&\quad \times P_{2n}^{(\lambda+1/2,\lambda-1/2)}(y)P_{2n'}^{(\lambda+1/2,\lambda-1/2)}(y)\Bigg\} \\
&= 2|C|^2\delta_{ll'}\delta_{mm'}\,\frac{\Gamma(2n+\lambda+1/2)\Gamma(2n+\lambda+3/2)}{(4n+2\lambda+1)(2n)!\,\Gamma(2n+2\lambda+1)} \\
&\quad \times \left\{|T_n(\tau)|^2 + k_n^2\left|\int_0^\tau T_n\,d\hat{\tau}\right|^2\right\}\delta_{nn'} \\
&= \delta_{nn'}\delta_{ll'}\delta_{mm'}
\end{aligned} \tag{18}$$

3.3. The Open Case $a = -1$

Here $r = \sinh s$ and

$$A_{lk} = ix^{1/2}(1-x)^\lambda F(\lambda + \tfrac{1}{2} + ik, \lambda + \tfrac{1}{2} - ik; \lambda + \tfrac{1}{2}; 1 - x)$$

with $x = (\cosh s + 1)/2$ (Zecca, 1996). By using the derivation and the recurrence relation properties of the hypergeometric function, one gets

$$\begin{aligned}
P^0_{klm} &\cong S^{(1)}_{lm}\,\frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)}\,(x^2 - x)^{\lambda/2} \\
&\quad \times \Bigg\{T_k x^{1/2} F\left(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik; \lambda + \frac{1}{2}; 1 - x\right) \\
&\quad - (x-1)^{1/2}\,\frac{k^2}{\lambda + 1/2}\,F\Bigg(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik;
\end{aligned}$$

$$\lambda + \frac{3}{2}; 1 - x\biggr) \int_0^\tau d\tau\ T_k\biggr\}$$

$$P^1_{klm} \cong S^{(2)}_{lm} \frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)} (x^2 - x)^{\lambda/2}$$

$$\times \biggl\{-T_k x^{1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik; \lambda + \frac{1}{2}; 1 - x\biggr)$$

$$- (x - 1)^{1/2} \frac{k^2}{\lambda + 1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik;$$

$$\lambda + \frac{3}{2}; 1 - x\biggr) \int_0^\tau d\tau\ T_k\biggr\}$$

$$\overline{Q}^1_{klm} \cong S^{(1)}_{lm} \frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)} (x^2 - x)^{\lambda/2}$$

$$\times \biggl\{-T_k x^{1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik; \lambda + \frac{1}{2}; 1 - x\biggr)$$

$$- (x - 1)^{1/2} \frac{k^2}{\lambda + 1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik;$$

$$\lambda + \frac{3}{2}; 1 - x\biggr) \int_0^\tau d\hat{\tau}\ T_k\biggr\}$$

$$\overline{Q}^0_{klm} \cong S^{(2)}_{lm} \frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)} (x^2 - x)^{\lambda/2}$$

$$\times \biggl\{-T_k x^{1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik; \lambda + \frac{1}{2}; 1 - x\biggr)$$

$$+ (x - 1)^{1/2} \frac{k^2}{\lambda + 1/2} F\biggl(\lambda + \frac{1}{2} + ik, \lambda + \frac{1}{2} - ik;$$

$$\lambda + \frac{3}{2}; 1 - x\biggr) \int_0^\tau d\hat{\tau}\ T_k\biggr\} \tag{19}$$

One has to distinguish between the two possible kinds of value of λ. *Consider first the subcase of integer* λ: $a = -1$, $\lambda = l + 1$ ($l = 0, 1, 2, 3 \ldots$).

From the differentiation formula and elementary cases of the hypergeometric function and the assumption on λ one gets from (19)

$$
\begin{aligned}
P^{0}_{klm} &= DS^{(1)}_{lm}\,\frac{\exp(i\mu_\star\sqrt{2}t}{rR^{3/2}(t)}\left\{T_k(\tau)q_\lambda(s,k) - kp_\lambda(s,k)\int_0^\tau d\hat{\tau}\; T_k\right\}\\
P^{1}_{klm} &= DS^{(2)}_{lm}\,\frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)}\left\{-T_k(\tau)q_\lambda(s,k) - kp_\lambda(s,k)\int_0^\tau d\hat{\tau}\; T_k\right\} \quad (20)\\
\overline{Q}^{1}_{klm} &= DS^{(1)}_{lm}\,\frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)}\left\{-T_k(\tau)q_\lambda(s,k) - kp_\lambda(s,k)\int_0^\tau d\hat{\tau}\; T_k\right\}\\
\overline{Q}^{0}_{klm} &= DS^{(2)}_{lm}\,\frac{\exp(i\mu_\star\sqrt{2}t)}{rR^{3/2}(t)}\left\{-T_k(\tau)q_\lambda(s,k) + kp_\lambda(s,k)\int_0^\tau d\hat{\tau}\; T_k\right\}
\end{aligned}
$$

where we have set

$$
D = \left\{\pi\left[k^2+\left(\frac{1}{2}\right)^2\right]\left[k^2+\left(\frac{3}{2}\right)^2\right]\cdots\left[k^2+\left(\lambda-\frac{1}{2}\right)^2\right]\right\}^{-1/2} \quad (21)
$$

$$
q_\lambda(s,k) = (\sinh s)^\lambda \cosh\frac{s}{2}\left(\frac{d}{d\cosh s}\right)^\lambda \frac{\cos ks}{\cosh(s/2)} \quad (22)
$$

$$
p_\lambda(s,k) = (\sinh s)^\lambda \sinh\frac{s}{2}\left(\frac{d}{d\cosh s}\right)^\lambda \frac{\sin ks}{\sinh(s/2)} \qquad (\lambda = l+1) \quad (23)
$$

The functions q_λ, p_λ satisfy a suitable set of recurrence, integral, and differential relations as shown in the Appendix. By means of the result in the Appendix one has

$$
\begin{aligned}
(\psi_{klm}, \psi_{k'l'm'}) &= 2|D|^2\delta_{ll'}\delta_{mm'}\left\{T_k\overline{T}_{k'}\int_0^\infty ds\; q_\lambda(s,k)q_\lambda(s,k')\right.\\
&\quad \left. + kk'\int_0^\tau d\hat{\tau}\; T_k\int_0^\tau d\hat{\tau}\; \overline{T}_{k'}\int_0^\infty ds\; p_\lambda(s,k)p_\lambda(s,k')\right\}\\
&= 2|D|^2\left[k^2+\left(\lambda-\frac{1}{2}\right)^2\right]\left[k^2+\left(\lambda-\frac{3}{2}\right)^2\right]\cdots\left[k^2+\left(\frac{1}{2}\right)^2\right]\\
&\quad \times\left(|T_k(\tau)|^2 + k^2\left|\int_0^\tau T_k(\hat{\tau})\,d\hat{\tau}\right|^2\right)
\end{aligned}
$$

$$\times \int_0^\infty ds\ q_0(s, k) q_0(s, k') \delta_{ll'} \delta_{mm'}$$
$$= \delta_{ll'} \delta_{mm'} \delta(k - k') \tag{24}$$

Consider now the subcase of half-integer λ: $a = -1$, $\lambda = l + \frac{1}{2}(l = 0, 1, 2, 3, \ldots)$. From the properties of the hypergeometric function and from (19) one gets

$$\begin{aligned}
P^0_{klm} &= ES^{(1)}_{lm} \frac{\exp(i\mu_\star \sqrt{2}\bar{t})}{rR^{3/2}(t)} \left\{ T_k(\tau) f_l(s, k) - k^2 g_l(s, k) \int_0^\tau d\hat{\tau}\ T_k \right\} \\
P^1_{klm} &= ES^{(2)}_{lm} \frac{\exp(i\mu_\star \sqrt{2}\bar{t})}{rR^{3/2}(t)} \left\{ -T_k(\tau) f_l(s, k) - k^2 g_l(s, k) \int_0^\tau d\hat{\tau}\ T_k \right\} \\
\bar{Q}^1_{klm} &= ES^{(1)}_{lm} \frac{\exp(i\mu_\star \sqrt{2}\bar{t})}{rR^{3/2}(t)} \left\{ -T_k(\tau) f_l(s, k) - k^2 g_l(s, k) \int_0^\tau d\hat{\tau}\ T_k \right\} \\
\bar{Q}^0_{klm} &= ES^{(2)}_{lm} \frac{\exp(i\mu_\star \sqrt{2}\bar{t})}{rR^{3/2}(t)} \left\{ -\ T_k(\tau) f_l(s, k) + k^2 g_l(s, k) \int_0^\tau d\hat{\tau}\ T_k \right\}
\end{aligned} \tag{25}$$

where the functions f_l, g_l and the constant E have been defined by

$$f_l(s, k) = (\sinh s)^{l+1/2} \cosh \frac{s}{2} \left(\frac{d}{d \cosh s} \right)^l F\left(1 + ik, 1 - ik; 1; -\sinh^2 \frac{s}{2} \right) \tag{26}$$

$$g_l(s, k) = (\sinh s)^{l+1/2} \sinh \frac{s}{2} \left(\frac{d}{d \cosh s} \right)^l F\left(1 + ik, 1 - ik; 2; -\sinh^2 \frac{s}{2} \right) \tag{27}$$

$$E = \left(\frac{2k}{\tanh \pi k} \right)^{1/2} \{[k^2 + l^2][k^2 + (l-1)^2] \ldots [k^2 + 1]\}^{-1/2} \tag{28}$$

From the results in the Appendix concerning the functions f_l, g_l one has

$$\begin{aligned}
(\psi_{klm}, \psi_{k'l'm'}) &= 2|E|^2 \delta_{ll'} \delta_{mm'} \left\{ T_k \bar{T}_{k'} \int_0^\infty ds\ f_l(s, k) f_l(s, k') \right. \\
&\quad \left. + kk' \int_0^\tau d\hat{\tau}\ T_k \int_0^\tau d\hat{\tau}\ \bar{T}_{k'} \int_0^\infty ds\ g_l(s, k) g_l(s, k') \right\} \\
&= 2|E|^2 [k^2 + l^2][k^2 + (l-1)^2] \ldots [k^2 + 1] \\
&\quad \times \left(|T_k(\tau)|^2 + k^2 \left| \int_0^\tau T_k(\hat{\tau})\ d\hat{\tau} \right|^2 \right)
\end{aligned} \tag{29}$$

$$\times \int_0^\infty ds\, f_0(s, k) f_0(s, k') \delta_{ll'} \delta_{mm'}$$
$$= \delta_{ll'} \delta_{mm'} \delta(k - k')$$

APPENDIX

The functions $q_\lambda(s, k)$ defined in equation (22) satisfy the following properties (compare with Dolginov and Toptygin, 1960; Bander and Itzykson, 1966)

$$q_{\lambda+1} = \left[\frac{d}{ds} - \lambda \coth s - \frac{1}{2} \tanh \frac{s}{2}\right] q_\lambda \tag{A1}$$

$$\left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right] q_{\lambda-1} = -\left[\frac{d}{ds} + (\lambda - 1) \coth s + \frac{1}{2} \tanh \frac{s}{2}\right] q_\lambda \tag{A2}$$

$$\left[\frac{d^2}{ds^2} + k^2 - \frac{\lambda(\lambda - 1)}{\sinh^2 s} + \frac{\lambda}{2\cosh^2 s/2}\right] q_\lambda = 0 \tag{A3}$$

To prove these relations, it suffices to prove that equation (A3) holds, because (A1) is a direct consequence of the definition (22), (A1) + (A2) imply (A3), and (A1) + (A3) imply (A2). To that end, by setting $q_\lambda = u(s)(\sinh s)^\lambda \cosh(s/2)$ in equation (A3) and then $t = (1 - \cosh s)/2 = -\sinh^2(s/2)$ in the resulting equation for u, one gets the equation

$$t(1 - t)\frac{d^2u}{dt^2} + \left[\lambda + \frac{1}{2} - (2\lambda + 2)t\right]\frac{du}{dt} - \left[k^2 + \left(\lambda + \frac{1}{2}\right)^2\right]u = 0 \tag{A4}$$

the solution of which is therefore

$$\begin{aligned} u &= F\left(\lambda + \frac{1}{2} + ik; \lambda + \frac{1}{2} - ik; \lambda + \frac{1}{2}; t\right) \\ &\cong \left(\frac{d}{dt}\right)^\lambda F\left(\frac{1}{2} + ik; \frac{1}{2} - ik; \frac{1}{2}; -\sinh^2 \frac{s}{2}\right) \\ &\cong \left(\frac{d}{d \cosh s}\right)^\lambda \left(\cos ks / \cosh \frac{s}{2}\right) \end{aligned} \tag{A5}$$

Using the recurrence relations just proved, and integrating by parts, one gets also

$$\begin{aligned}
&\int_0^\infty ds\, q_\lambda(s, k) q_\lambda(s, k') \\
&= \int_0^\infty ds\, q_\lambda(s, k) \left[\frac{d}{ds} - (\lambda - 1) \coth s - \frac{1}{2} \tanh \frac{s}{2}\right] q_{\lambda-1}(s, k') \\
&= \int_0^\infty ds\, q_{\lambda-1}(s, k') \left[-\frac{d}{ds} - (\lambda - 1) \coth s - \frac{1}{2} \tanh \frac{s}{2}\right] q_\lambda(s, k) \\
&= \left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right] \int_0^\infty ds\, q_{\lambda-1}(s, k) q_{\lambda-1}(s, k') \\
&= \left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right]\left[k^2 + \left(\lambda - \frac{3}{2}\right)^2\right] \ldots \\
&\quad \times \left[k^2 + \left(\frac{1}{2}\right)^2\right] \int_0^\infty ds \cos ks \cos k's \\
&= \left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right]\left[k^2 + \left(\lambda - \frac{3}{2}\right)^2\right] \ldots \\
&\quad \times \left[k^2 + \left(\frac{1}{2}\right)^2\right] \frac{\pi}{2} \delta(k - k')
\end{aligned} \tag{A6}$$

By similar procedures one can show that the functions $p_\lambda(s, k)$ defined in (23) satisfy the properties

$$p_{\lambda+1} = \left[\frac{d}{ds} - \lambda \coth s - \frac{1}{2} \coth \frac{s}{2}\right] p_\lambda \tag{A7}$$

$$\left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right] p_{\lambda-1} = -\left[\frac{d}{ds} + (\lambda - 1) \coth s + \frac{1}{2} \coth \frac{s}{2}\right] p_\lambda \tag{A8}$$

$$\left[\frac{d^2}{ds^2} + k^2 - \frac{\lambda(\lambda - 1)}{\sinh^2 s} - \frac{\lambda}{2 \sinh^2 s/2}\right] p_\lambda = 0 \tag{A9}$$

and that their integral property is the same as that of the $q_\lambda(s, k)$:

$$\begin{aligned}
&\int_0^\infty ds\, p_\lambda(s, k) p_\lambda(s, k') \\
&= \left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right]\left[k^2 + \left(\lambda - \frac{3}{2}\right)^2\right] \ldots \left[k^2 + \left(\frac{1}{2}\right)^2\right] \int_0^\infty ds \sin ks \sin k's
\end{aligned} \tag{A10}$$

$$= \left[k^2 + \left(\lambda - \frac{1}{2}\right)^2\right]\left[k^2 + \left(\lambda - \frac{3}{2}\right)^2\right] \ldots \left[k^2 + \left(\frac{1}{2}\right)^2\right]\frac{\pi}{2}\delta(k - k') \qquad \text{(A11)}$$

The previous procedures can be extended to prove that the functions f_l, g_l defined in equations (26) and (27) have the properties

$$f_{l+1} = \left[\frac{d}{ds} - \left(l + \frac{1}{2}\right)\coth s - \frac{1}{2}\tanh\frac{s}{2}\right]f_l \qquad \text{(A12)}$$

$$[k^2 + l^2]\, f_{l-1} = -\left[\frac{d}{ds} + \left(l - \frac{1}{2}\right)\coth s + \frac{1}{2}\tanh\frac{s}{2}\right]f_l \qquad \text{(A13)}$$

$$\left[\frac{d^2}{ds^2} + k^2 - \frac{l^2 - 1/4}{\sinh^2 s} + \frac{2l + 1}{4\cosh^2(s/2)}\right]f_l = 0 \qquad \text{(A14)}$$

and

$$g_{l+1} = \left[\frac{d}{ds} - \left(l + \frac{1}{2}\right)\coth s - \frac{1}{2}\coth\frac{s}{2}\right]g_l \qquad \text{(A15)}$$

$$[k^2 + l^2]\, g_{1-1} = -\left[\frac{d}{ds} + \left(l - \frac{1}{2}\right)\coth s + \frac{1}{2}\coth\frac{s}{2}\right]g_l \qquad \text{(A16)}$$

$$\left[\frac{d^2}{ds^2} + k^2 - \frac{l^2 - 1/4}{\sinh^2 s} - \frac{2l + 1}{4\sinh^2(s/2)}\right]g_l = 0 \qquad \text{(A17)}$$

and

$$\int_0^\infty ds\, f_l(s, k) f_l(s, k') = [k^2 + l^2][k^2 + (l-1)^2] \ldots [k^2 + 1] \times \int_0^\infty ds\, f_0(s, k) f_0(s, k') \qquad \text{(A18)}$$

$$\int_0^\infty ds\, g_l(s, k) g_l(s, k') = [k^2 + l^2][k^2 + (l-1)^2] \ldots [k^2 + 1] \times \int_0^\infty ds\, g_0(s, k) g_0(s, k') \qquad \text{(A19)}$$

By using a linear transformation formula and a differentiation formula for the hypergeometric function one has

$$\int_0^\infty ds\, f_0(s, k) f_0(s, k') = \frac{1}{k^2} \int_0^\infty ds\, g_0(s, k) g_0(s, k') \tag{A20}$$

From equation (A14) one gets

$$(k^2 - k'^2) \int_0^a ds\, f_0(s, k) f_0(s, k')$$
$$= [f_0'(s, k) f_0(s, k') - f_0'(s, k') f_0(s, k)]_0^a \tag{A21}$$

and finally, from the asymptotic behavior of the hypergeometric function one gets

$$\int_0^a ds\, f_0(s, k) f_0(s, k') \to 2 \left| \frac{\Gamma(2ik)}{\Gamma(1 + ik)\Gamma(ik)} \right|^2 \frac{\sin(k' - k)a}{k' - k}$$
$$\to (2k)^{-1} \tanh \pi k\, \delta(k' - k) \tag{A22}$$

for $a \to \infty$.

REFERENCES

Abramovitz, W., and Stegun I. E. (1970). *Handbook of Mathematical Functions,* Dover, New York.

Arfken G. B., and Weber, H. J. (1995). *Mathematical Methods for Physicists,* Academic Press, San Diego, California.

Bander, M., and Itzykson, C. (1966). *Reviews of Modern Physics,* **38**, 346.

Birrell, N. D., and Davies, P. C. W. (1982). *Quantum Fields in Curved Space-Time,* Cambridge University Press, Cambridge.

Buchdahl, H. A. (1962). *Nuovo Cimento,* **25**, 486.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes,* Oxford University Press, Oxford.

Dolginov, A. Z., and Toptygin, I. N. (1960). *Soviet Physics JETP,* **37**, 1022.

Ford, L. H. (1976). *Physical Review D,* **14**, 3304.

Fulling, S. A. (1973). *Physical Review D,* **7**, 2850.

Fulling, S. A. (1989). *Aspects of Quantum Field Theory in Curved Space-Time,* Cambridge University Press, Cambridge.

Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space Time,* Cambridge University Press, Cambridge.

Hu, B. L. (1974). *Physical Review D,* **9**, 3263.

Illge, R. (1993). *Communications in Mathematical Physics,* **158**, 433.

Kolb, E. W., and Turner, M. S. (1990). *The Early Universe,* Addison-Wesley, New York.

Montaldi, E. and Zecca, A. (1994). *International Journal of Theoretical Physics,* **33**, 1053.

Newman, E. T., and Penrose, R. (1962). *Journal of Mathematical Physics,* **3**, 566.

Parker, L. (1969). *Physical Review D,* **183**, 1057.
Parker, L. (1971). *Physical Review D,* **3**, 346.
Parker, L., and Fulling, S. A. (1974). *Physical Review D,* **9**, 341.
Penrose, R., and Rindler, W. (1990). *Spinors and Space-Time,* Cambridge University Press, Cambridge.
Unruh, W. G. (1974). *Physical Review D,* **10**, 3194.
Wünsch, V. (1978). *Beiträge zur Analysis,* **12**, 47.
Wünsch, V. (1979). *Beiträge zur Analysis* **13**, 147.
Zecca, A. (1996). *Journal of Mathematical Physics,* **37**, 874.
Zecca, A. (1997). *International Journal of Theoretical Physics,* **36,** 1413.